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REGULARIZATION GROUP AND COMPLETENESS OF FIELD THEORIES. NUCLEON-
COUPLINGS.

By

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A B S T R A C T

The problem of the mathematical consistency of a field theory defined by a given local Hamiltonian is studied in terms of the propagator (Green's function) formalism.

It is necessary for the mathematical consistency of a theory that all the branching equations satisfied by its propagators be covariant under the transformations of its renormalization group (which can be explicitly written). This analysis (which differs in method, but not in principle, from the standard renormalization program) permits to find systematically and explicitly all the terms that need be added to the original Hamiltonian if this was not complete to start with, i.e. if covariance could not be secured for the set of branching equations obtained from it alone.

Local non-renormalizable theories are mathematically meaningless, because they originate from only fragments of Hamiltonians which are meaningful only if taken as wholes; the missing terms (even if infinite in number) can be exactly reconstructed with the present method, which leads naturally to identify the concepts of mathematical consistency and of physical completeness. All meaningful relations among coupling constants, such as symmetry requirements, must remain invariant under the renormalization group, which plays a rôle as important in the search for completeness, as that of the gauge group in electrodynamical problems.

For the sake of concreteness, and as a first example, this method is illustrated with reference to the study of the standard meson-nucleon couplings, scalar and pseudoscalar, neutral and charged; the well known φ^3 and φ^4 (scalar), φ^4 (pseudoscalar) terms are obtained (a precedent erroneous statement about the renormalizability of the neutral scalar coupling is corrected, so that now all results agree with the expected ones). Another example is treated in the Appendix.

I. - Introduction.

1. In previous works⁽¹⁾ a rigorous theory of renormalization was developed, in which ultraviolet divergences were recognized to be a phenomenon typical of hyperbolic equations and well known to mathematicians since a long time⁽²⁾; the replacement of ordinary integrals with suitable generalization of Hadamard's finite-part integrals was shown to be all that is required to perform correctly the transition from the unrenormalized to the renormalized theory. No u.v. infinities or ambiguities can ever occur with this method, no writing of counter-terms is required, in particular the introduction of "proper" self-energy or vertex parts (which destroy the linearity of the theory) is totally avoided.

What happens is simply that divergent quantities are replaced with indeterminate ones, which are fully displayed (as is done with the arbitrary constants of distribution theory, which is comprised within the scope of our work); it is then required that such indetermination be of no physical consequence, exactly as is the case for the gauge in electrodynamics.

This is achieved by imposing, first of all, that renormalization (bars denote renormalized quantities; $\lambda_1, \lambda_2, \dots, \lambda_K$ are the parameters of the given theory: masses and coupling constants) should not change the values of the elements of the U matrix:

$$(1) \quad M_{FI}(\lambda_1, \dots, \lambda_K) = \bar{M}_{FI}(\bar{\lambda}_1, \dots, \bar{\lambda}_K)$$

This criterion, which is the keystone of Dyson's⁽³⁾ treatment of the subject, permits, through an appropriate use of combinatorial techniques, to derive the differential equations of the transformation

which changes λ_i into $\bar{\lambda}_i$; since this transformation changes if the finite-part integration prescription is changed, we may speak of "renormalization group" (under the change of prescription, i.e. of the indeterminate quantities mentioned before). To secure that no physical consequences are caused by this indetermination one must require, indeed, that all the equations among propagation kernels (which fully define the field theory) be covariant under the renormalization group: This condition suffices to derive the differential equations of the transformations. Our analysis gives also equations for the so-called "wave-function" renormalization constants, which fully determine them; no such constants are instead required for "vertex parts" of various specifications (the latter constants, which are typical of the standard treatment, could be trivially computed a posteriori with our method, but are of no interest for our purposes).

As a result, it was found that Dyson's heuristic conditions for the renormalizability of a theory are fully confirmed as necessary conditions by our investigation in the case of the so-called "renormalizable" theories. One discrepancy which was found, regarding the neutral scalar meson-nucleon coupling, which was judged to be non-renormalizable because it did not seem legitimate at the time to assume the commutability of some non symmetrical finite-part integrations, was recognized later to be due to an excess of caution, and is removed here. (The proof of the legitimacy of commuting f.p. integrals also in the case of non symmetric integrands is given in a forthcoming paper by B. Preziosi). The investigation of the situation that arises with "non-renormalizable theories" (according to Dyson's rules) will come next in our program; it may yield some unexpected results because, for instance, the Pauli principle causes

manifestly tremendous cancellations among the divergent contributions.

The fundamental requirement (1) amounts to imposing that, if any number of quantities Q_n (to be measured experimentally) are calculated with, say, two different f.p. integration prescriptions, one finds

$$(2) \quad Q_n = f_n(\lambda_1, \dots, \lambda_K) = \bar{f}_n(\bar{\lambda}_1, \dots, \bar{\lambda}_K)$$

Suppose that the first K of the Q_n are functionally independent, so that

$$(3) \quad \begin{aligned} \lambda_i &= \lambda_i(Q_1, \dots, Q_K) \\ \bar{\lambda}_i &= \bar{\lambda}_i(Q_1, \dots, Q_K) \end{aligned} ;$$

then the consistency requirement among physical quantities is automatically satisfied:

$$(4) \quad \begin{aligned} Q_{K+2} &= f_{K+2}(\lambda_1(Q_1, \dots, Q_K), \dots, \lambda_K(Q_1, \dots, Q_K)) = \\ &= \bar{f}_{K+2}(\bar{\lambda}_1(Q_1, \dots, Q_K), \dots, \bar{\lambda}_K(Q_1, \dots, Q_K)) \end{aligned}$$

Our procedure secures that (1) is satisfied, and therefore (2) and (4), if and only if the theory is "renormalizable" - or, as we may better say, consistent or "complete". While, say, dispersion relations work directly on relations of type (4), we have a parametrization $\lambda_1, \dots, \lambda_K$ which does not favor any special quantity Q . It is

apparent that $\lambda_1 \dots \lambda_k$ are not to be considered as having the experimental values of the quantities which they formally represent in the ordinary fashion of writing Hamiltonians, but as indeterminate parameters to be fixed a posteriori through relations (3); prescriptions may be found for which $\lambda_1 = 2, \dots, \lambda_k = 2$ but this question is not of interest in the present connection.

2.- An interesting feature of this method is that it permits to assess, in a perfectly straightforward manner, by the simple use of combinatorics and without the need of performing actual integrations, which additional terms must be added to the Hamiltonian to render it complete if it was not such to start with. This comes about most simply, because if those terms (the exact nature of which can be recognized immediately) are not included, i.e. if, equivalently, the respective new branching equations are not considered, then it is impossible to secure the wanted covariance of the field equations under the renormalization group.

This is not different, in principle, from the standard search of such terms which is made in order to compensate divergences with counter terms; apart from questions of rigor, however, the techniques that our method requires permit to deal with any situation in a fully automatic and consistent manner, which is always the same and will allow the immediate recognition of all possible interferences among graphs⁽⁴⁾ in situations, such as with Fermi interactions, which would be otherwise hopeless.

The present work deals with the case of PS and S, charged and neutral, mesons coupled to the nucleon field. We have chosen a thoroughly familiar case, in order to exhibit the technique at its simplest. The results which we find are the well-known ones, that renormalizability (for us, completeness) requires the addition of terms $\lambda_4 \varphi^4$ and $\lambda_3 \varphi^3 + \lambda_4 \varphi^4$ respectively, to the interaction.⁽⁵⁾

This should serve, in our intention, mainly as an example to which we may refer in future in the study of theories which are as yet unexplored. The results reported here, however, will be of interest also from a different point of view, that of quantitative analysis: the coefficients of the differential equations of our renormalizing transformations can in fact be actually computed, with as much approximation as is wanted, and this will permit to study the solutions of these equations in a rigorous, even if approximate, way.

It should be clear that the knowledge of the group properties at large, and not only locally, may open the way to a deeper understanding of the structure of the theory, e.g. as regards mass spectra, or parameter quantization (if any such properties exist in the theory). Work in this direction is in progress, and will be reported on in due time.

In the Appendix we give another example of the application of this technique: given two meson fields φ_1 and φ_2 with coupling $\lambda_1 \varphi_1' \varphi_1^2$, the theory is incomplete unless the terms $\lambda_1 \varphi_1'^4$ and $\lambda_2 \varphi_2'^4$ are added to the Hamiltonian. Numerical values for all such constants cannot be given on the basis of the present, purely combinatorial, analysis, but the exact form of the terms is immediately found.

II. - Neutral P.S. Coupling.

We use throughout in this work the notation and the results contained in our previous works on the subject. Let us first try to determine the necessary conditions for ^{the}renormalizability of the pseudo-scalar theory. If the Hamiltonian contains only the coupling terms:

$$(5) \quad H = g \bar{\psi} \gamma^5 \psi \varphi$$

We obtain the following expressions for the kernels and their derivatives with respect to $\lambda = -ig$:

$$(6) \quad K_{N_0 P_0} = K(x_1 \dots x_{N_0} | y_1 \dots y_{N_0} | t_1 \dots t_{P_0}) = \sum_{N(P_0)} \frac{A^N}{N!} \int d\tau_1 \dots \int d\tau_N \delta_5^1 \dots \delta_5^N (x_1 \dots x_{N_0} \tau_1 \dots \tau_N | y_1 \dots y_{N_0} \tau_1 \dots \tau_N) [t_1 \dots t_{P_0} \tau_1 \dots \tau_N]$$

$$(7) \quad \frac{\partial K_{N_0 P_0}}{\partial \lambda} = \int d\tau \delta_5^1 K(x_1 \dots x_{N_0} \tau | y_1 \dots y_{N_0} \tau | t_1 \dots t_{P_0})$$

The last equation when we separate the divergent from the convergent part can be written (see II-3):

$$(8) \quad \frac{\partial K_{N_0 P_0}}{\partial \lambda} = \int d\tau \delta_5^1 K^{(1)}(x_1 \dots x_{N_0} \tau | y_1 \dots y_{N_0} \tau | t_1 \dots t_{P_0}) + \sum_{S, \sigma \neq 0} \left(\frac{\partial K_{N_0 P_0}}{\partial \lambda} \right)_{S, \sigma}^{(1)}$$

or, introducing the "divergent cores"

$$(9) \quad C^{(1)}(\tilde{u}_1 \dots \tilde{u}_S | \tilde{u}_1' \dots \tilde{u}_\sigma') = \int d\tau \delta_5^1 \sum_{N, H} \frac{A^N}{N!} \mathcal{D}_2^{N-1} \delta_5^1 \dots \delta_5^N (\tilde{u}_1 \dots \tilde{u}_S \tau_1 \dots \tau_N | \tilde{u}_1' \dots \tilde{u}_\sigma' \tau_1 \dots \tau_N)$$

$$(10) \quad \frac{\partial K_{N_0 P_0}}{\partial \lambda} = \int d\tau \delta_5^1 K^{(1)}(x_1 \dots x_{N_0} \tau | y_1 \dots y_{N_0} \tau | t_1 \dots t_{P_0}) + \left(C_{0,0}^{(1)} + \frac{P_0}{2} C_{0,2}^{(1)} + N_0 C_{1,0}^{(1)} \right) K_{N_0 P_0}^{(1)} + C_{0,2}^{(2)} \frac{\partial K_{N_0 P_0}}{\partial m_b^2} + C_{1,0}^{(2)} \frac{\partial K_{N_0 P_0}}{\partial m_f} + \left(\frac{1}{2} \lambda C_{0,2}^{(1)} + \lambda C_{1,0}^{(1)} + C_{1,1}^{(1)} \right) \frac{\partial K_{N_0 P_0}}{\partial \lambda} + C_{0,1}^{(1)} \int d\tau K(x_1 \dots x_{N_0} \tau | y_1 \dots y_{N_0} \tau | t_1 \dots t_{P_0})$$

The equation (10)¹⁾ is found by observing that it suffices to consider in (8) only the terms which satisfy the relation:

$$(11) \quad 3S + \sigma \leq 4$$

that the terms with $S=0$, $\sigma=1,3$ are zero because of Furry's

¹⁾ If we compare this expression with (II-26) we find that all but the last term of (10) are contained in (II-26). This term comes from $\left(\frac{\partial K}{\partial \lambda} \right)_{0,0}^{(1)}$ which is not zero in the present theory.

theorem and the *Cotes* satisfy the conditions (which are easily proved by actual computation):

$$\begin{aligned}
 (12) \quad C^{(\lambda)}(x y) &= C_{0,2}^{(1)\lambda} [x y] + C_{0,2}^{(2)\lambda} \frac{\partial [x y]}{\partial m_h^2} \\
 C^{(\lambda)}\left(\frac{x}{y}\right) &= C_{1,0}^{(1)\lambda} (x y) + C_{1,0}^{(2)\lambda} \frac{\partial (x y)}{\partial m_f} \\
 C^{(\lambda)}\left(\frac{x}{y} \middle| \frac{z}{t}\right) &= C_{1,1}^{(1)\lambda} \int d\tilde{z} (x z) y^2 (x y) [\tilde{z} t] \\
 C^{(\lambda)}(x y z t) &= C_{0,4}^{(1)\lambda} \int d\tilde{z} [x y z t \tilde{z} \tilde{z} \tilde{z} \tilde{z}]
 \end{aligned}$$

We can now transform the kernel K_{Nop_0} as follows:

$$K_{Nop_0}(\lambda, m_f, m_h^2) = A_{Nop_0}(\lambda, m_f, m_h^2) \tilde{K}_{Nop_0}(\lambda, m_f, m_h^2)$$

and put:

$$\begin{aligned}
 \mathcal{D} &= c^\lambda \frac{\partial}{\partial \lambda} - C_{0,2}^{(2)\lambda} \frac{\partial}{\partial m_h^2} - C_{1,0}^{(2)\lambda} \frac{\partial}{\partial m_f} \\
 (13) \quad c^\lambda &= 1 - \frac{1}{2} \lambda C_{0,2}^{(1)\lambda} - \lambda C_{1,0}^{(1)\lambda} - C_{1,1}^{(1)\lambda}
 \end{aligned}$$

$$A_{Nop_0} = A z_2^{N_0} z_3^{\frac{P_2}{2}}$$

Then following the same pattern as that of work II

from the (10) we obtain the equations:

$$(14) \quad \mathcal{D}(g A) = C_{00}^{\lambda} \quad ; \quad \mathcal{D}(g z_1) = C_{11}^{(1)\lambda} \quad ; \quad \mathcal{D}(g z_3) = C_{02}^{(1)\lambda}$$

$$z_2^{-1} z_3^{-\frac{1}{2}} \mathcal{D} \tilde{K}_{N,p_0} = \int d\gamma \gamma_5^3 \tilde{K}(\gamma_1 \dots \gamma_N / t_1 \dots t_{p_0}) + C_{04} z_2^{-1} z_3^{\frac{3}{2}} \int dZ \tilde{K}(\gamma_1 \dots \gamma_N / t_1 \dots t_{p_0})$$

A change of parameters:

$$\lambda = \lambda (\bar{\lambda} \bar{m}_f \bar{m}_n^2)$$

$$m_f = m_f (\bar{\lambda} \bar{m}_f \bar{m}_n^2)$$

$$m_n^2 = m_n^2 (\bar{\lambda} \bar{m}_f \bar{m}_n^2)$$

so that :

$$(1) \quad \mathcal{D}(\bar{m}_f) = 0 \quad \mathcal{D}(\bar{m}_n^2) = 0 \quad \mathcal{D}(\bar{\lambda}) = z_2 z_3^{\frac{1}{2}}$$

gives :

$$\tilde{K}(\lambda m_f m_n^2) = \bar{K}(\bar{\lambda} \bar{m}_f \bar{m}_n^2)$$

and transforms the last equation of (14) in

$$(15) \quad \frac{2}{\lambda} \tilde{K}_{N,p_0} = \int d\gamma \gamma_5^3 \bar{K}(\gamma_1 \dots \gamma_N / t_1 \dots t_{p_0}) + C_{04} z_3^{\frac{3}{2}} z_2^{-1} \int dZ \bar{K}(\gamma_1 \dots \gamma_N / t_1 \dots t_{p_0})$$

If we compare the last equation with equation (7) we have to conclude

that the pseudoscalar theory, formulated in function of only three parameters, ^(\lambda m_f m_n^2) does not satisfy the first condition of renormalizability,

This condition in fact requires $\left(\frac{\partial}{\partial \lambda} \right)$ that the derivative of ^{the} Kernel \tilde{K} with respect to $\bar{\lambda}$ reduce to the formula:

$$\frac{2}{\lambda} \tilde{K}_{N,p_0} = \int d\gamma \gamma_5^3 \bar{K}(\gamma_1 \dots \gamma_N / t_1 \dots t_{p_0})$$

The Hamiltonian (5) which we have used is therefore not complete; we

must add other terms to it in order to render the pseudoscalar theory renormalizable.

2.- Comparing the equation (15) with (4) we deduce that it is necessary to add a new term to the Hamiltonian (5); it is easily seen that the interaction must change to:

$$(16) \quad H = g_1 \bar{\psi} \gamma_5 \psi + g_2 \psi^4$$

no other additional term being permissible.

We obtain from (16) the new kernel:

$$(17) \quad K_{N, P_0} = \sum_{S_1=0}^{\infty} \sum_{S_2=0}^{\infty} \frac{\lambda_1^{S_1}}{S_1!} \frac{\lambda_2^{S_2}}{S_2!} \int d\tau_1 \dots \int d\tau_N \gamma_5^{S_1} \dots \gamma_5^{S_2}$$

$$\left(\begin{matrix} x_1 \dots x_N, \tau_1 \dots \tau_N \\ y_1 \dots y_N, \tau_1 \dots \tau_N \end{matrix} \right) \left[t_1 \dots t_{P_0}, \tau_1 \dots \tau_{S_1}, \tau_{S_1+1}, \tau_{S_1+1}, \tau_{S_1+1}, \tau_{S_1+1}, \tau_{S_1+1}, \tau_{S_1+1} \dots \tau_N, \tau_N, \tau_N, \tau_N \right]$$

where

$$\lambda_1 = -ig_1$$

$$\lambda_2 = -ig_2$$

$$S_1 + S_2 = N$$

The first sum is over all indices S_1 having the same parity as P_0 .

. The derivatives ¹⁾ of the kernel with respect to the four parameters of the theory, coupling constants and masses, are

$$(18) \quad \frac{\partial K_{N_0 P_0}}{\partial \lambda} = \int d\{ \gamma_5 \} K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \{ t_1 \dots t_{P_0} \} \right)$$

$$(19) \quad \frac{\partial K_{N_0 P_0}}{\partial \lambda_4} = \int d\{ \gamma \} K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \{ t_1 \dots t_{P_0} \} \right)$$

$$(20) \quad \frac{\partial K_{N_0 P_0}}{\partial m_f} = -i \int d\{ \gamma \} K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \{ t_1 \dots t_{P_0} \} \right)$$

$$(21) \quad \frac{\partial K_{N_0 P_0}}{\partial m_0^2} = -\frac{i}{2} \int d\{ \gamma \} K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \{ t_1 \dots t_{P_0} \} \right)$$

1) The equations (18) (19) (20) (21) are correct as they stand if we put $\{ \gamma \} = f(m)$, $\{ \gamma \gamma \} = g(m)$; when instead we put $\{ \gamma \gamma \} = 0$, $\{ \gamma \gamma \gamma \} = 0$

3- In order to separate the convergent from the divergent part of the derivative of the kernel with respect to λ_1 we write:

$$(22) \quad \frac{\partial K_{n,p_0}}{\partial \lambda_1} = \sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{\lambda_1^{h_1}}{h_1!} \frac{\lambda_2^{h_2}}{h_2!} \int d\beta \, \gamma_5^3 \gamma_5' \dots \gamma_5^{h_1} \hat{D}^{h_1, h_2, \dots, 1}$$

$$K^{(4)} \left(\begin{array}{c} x_1 \dots x_{N_0} \, \beta \beta_1 \dots \beta_{h_1} \\ y_1 \dots y_{N_0} \, \beta \beta_1 \dots \beta_{h_1} \end{array} \right) \left(\beta \beta_1 \dots \beta_{h_1} \beta_{h_1+1} \beta_{h_1+2} \beta_{h_1+3} \beta_{h_1+4} \dots - t_1 \dots t_{h_2} \right)$$

(continuation of footnote 1) of the preceding page)

the derivative of the kernel with respect to m_0^2 is given by (2) only if all the expressions

$$(a) \quad \iint d\beta_1 d\beta_2 [\beta_1 \beta_2 \beta_3 \beta_4] K \left(\begin{array}{c} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{array} \middle| t_1 \dots t_{p_0} \beta_1 \beta_2 \right)$$

are taken to vanish (see ref. 6)

The convention $\langle \beta \beta \rangle = f(m)$, $[\beta \beta] = g(m)$ (or $\langle \beta \beta \rangle = 0$, $[\beta \beta] = 0$) affects only the renormalization coefficients. One can use either definition provided care is taken that the expression (a) vanishes if $[\beta \beta] = 0$

where the $\hat{\mathcal{D}}_i^{h_i-1}$ are defined as in (III).

From a generalization of formula (I-19) it follows :

$$\frac{\partial K}{\partial \lambda_1} = \sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{1}{h_1!} \frac{1}{h_2!} \int d\tau \gamma_5^{\tau} \dots \gamma_5^{\tau} \hat{\mathcal{D}}_i^{h_1+h_2-1}$$

$$(23) \quad \sum_{\tau=0}^{h_1+1} \sum_{\rho=0}^{h_2+1} \frac{1}{\tau!} \frac{1}{\rho!} \sum_{c_p} \sum_{c_2} \sum_{c_2} (-1)^{P(\tau, \rho)} \begin{pmatrix} \tau_{c_1} \dots \tau_{c_{\tau}} \\ \rho_{c_1} \dots \rho_{c_{\rho}} \end{pmatrix} [\tau_{c_1} \dots \tau_{c_{\tau}}]$$

$$K^{(1)} \left(\begin{matrix} \tau_{c_1} \dots \tau_{c_{\tau}} & x_1 \dots x_{N_0} \\ \rho_{c_1} \dots \rho_{c_{\rho}} & y_1 \dots y_{N_0} \end{matrix} \middle| \begin{matrix} \tau_{c_1} \dots \tau_{c_{\tau}} & t_1 \dots t_{p_0} \end{matrix} \right)$$

or, denoting with $\left(\frac{\partial K}{\partial \lambda} \right)_{s, \sigma}^{(1)}$ the contributions coming to from different values of s, σ :

$$(24) \quad \frac{\partial K}{\partial \lambda_1} = \int d\tau \gamma_5^{\tau} K^{(1)} \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \begin{matrix} t_1 \dots t_{p_0} \end{matrix} \right) + \sum_{s, \sigma \geq 0} \left(\frac{\partial K_{N_0, p_0}}{\partial \lambda_1} \right)_{s, \sigma}^{(1)}$$

where the sum is over all values s, σ satisfying the relation:

$$3s + \sigma \leq 4$$

The terms with $s=0$ $\sigma=1, 3$ are zero

We can define the "divergent cores":

$$(25) \quad e^{(\lambda_1)} \left(\begin{array}{c} \dot{u}_1 \dots \dot{u}_n \\ \dot{v}_1 \dots \dot{v}_n \end{array} \middle| \begin{array}{c} \dot{w}_1 \dots \dot{w}_m \end{array} \right) = \int d\zeta \, \gamma_5^? L \left(\begin{array}{c} \dot{u}_1 \dots \dot{u}_n \\ \dot{v}_1 \dots \dot{v}_n \end{array} \middle| \zeta \begin{array}{c} \dot{w}_1 \dots \dot{w}_m \end{array} \right)$$

where we have put :

$$(26) \quad L \left(\begin{array}{c} \dot{u}_1 \dots \dot{u}_n \\ \dot{v}_1 \dots \dot{v}_n \end{array} \middle| \zeta \begin{array}{c} \dot{w}_1 \dots \dot{w}_m \end{array} \right) = \sum_{h_1}^{\infty} \sum_{h_2}^{\infty} \frac{\lambda_1^{h_1}}{h_1!} \frac{\lambda_2^{h_2}}{h_2!} \mathcal{D}_\zeta^{h_1+h_2-1} \gamma_5^{?} \dots \gamma_5^{?}$$

$$\left(\begin{array}{c} \dot{u}_1 \dots \dot{u}_n \\ \dot{v}_1 \dots \dot{v}_n \end{array} \middle| \begin{array}{c} \zeta_1 \dots \zeta_{h_1} \\ \zeta_1 \dots \zeta_{h_1} \end{array} \right) \left[\begin{array}{c} \zeta_1 \dots \zeta_{h_1} \zeta_{h_1+1} \zeta_{h_1+1} \zeta_{h_1+1} \dots \zeta_{h_1+h_2-1} \\ \zeta_1 \dots \zeta_{h_1} \zeta_{h_1+1} \zeta_{h_1+1} \zeta_{h_1+1} \dots \zeta_{h_1+h_2-1} \end{array} \right] \begin{array}{c} \dot{w}_1 \dots \dot{w}_m \\ \dot{w}_1 \dots \dot{w}_m \end{array}$$

The sum is over all h_i with the same parity as $m+1$. By means of (25) and (I-26), (I-29), we obtain immediately

$$\left(\frac{\partial K}{\partial \lambda} \right)_{a_0}^{(3)} = C_{a_0} K_{N_0 p_0}$$

$$(27) \quad \left(\frac{\partial K}{\partial \lambda} \right)_{a_2}^{(3)} = \sum_{N_1} \sum_{N_2} \frac{\lambda_1^{N_1}}{N_1!} \frac{\lambda_2^{N_2}}{N_2!} \int d\zeta' \dots \int d\zeta'_{N_1+N_2} \gamma_5^{?} \dots \gamma_5^{? N_1}$$

$$\left(\begin{array}{c} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{array} \middle| \begin{array}{c} \zeta'_1 \dots \zeta'_{N_0} \\ \zeta'_1 \dots \zeta'_{N_0} \end{array} \right) \sum_{cr} e^{(\lambda_1)} \left(\begin{array}{c} \dot{u}_{m_1} \dots \dot{u}_{m_1} \\ \dot{u}_{m_2} \dots \dot{u}_{m_2} \end{array} \right) \left[\begin{array}{c} \dot{u}_{m_1} \dots \dot{u}_{m_1} \\ \dot{u}_{m_2} \dots \dot{u}_{m_2} \end{array} \right]$$

\sum_{cr} is the sum over all combinations of $p_0 + N_1 + 4N_2 = 2\gamma + 2$ in which

$$\left(\frac{\partial K}{\partial \lambda}\right)_{10}^{(2)} = \sum_{N_1} \sum_{N_2} \frac{\lambda_1^{N_1}}{N_1!} \frac{\lambda_2^{N_2}}{N_2!} \int d\lambda_1' \dots \int d\lambda_{N_1+N_2}' \gamma_5' \dots \gamma_5^{N_1} \sum_{CT} C \left(\begin{matrix} \lambda_1^{N_1} \\ \gamma_{m_1}' \dots \gamma_{m_{N_1}}' \end{matrix} \right) \left(\begin{matrix} \lambda_2^{N_2} \\ \gamma_{m_{N_1+1}}' \dots \gamma_{m_{N_1+N_2}}' \end{matrix} \right) X^{14} -$$

$$[t_1 \dots t_{p_0} \lambda_1' \dots \lambda_{N_1}' \lambda_{N_1+1}' \dots \lambda_{N_1+N_2}' \lambda_{N_1+N_2+1}' \dots \lambda_{N_1+N_2+1}' \dots]$$

$$(28) \left(\frac{\partial K}{\partial \lambda_1}\right)_{04}^{(3)} = \sum_{N_1} \sum_{N_2} \frac{\lambda_1^{N_1}}{N_1!} \frac{\lambda_2^{N_2}}{N_2!} \int d\lambda_1' \dots \int d\lambda_{N_1+N_2}' \gamma_5' \dots \gamma_5^{N_1} \left(\begin{matrix} \lambda_1^{N_1} \\ \gamma_{m_1}' \dots \gamma_{m_{N_1}}' \end{matrix} \right) \sum_{CT} C \left(\begin{matrix} \lambda_2^{N_2} \\ \gamma_{m_{N_1+1}}' \dots \gamma_{m_{N_1+N_2}}' \end{matrix} \right) \left[\begin{matrix} \lambda_1^{N_1} \\ \gamma_{m_1}' \dots \gamma_{m_{N_1}}' \end{matrix} \right]$$

$$\left(\frac{\partial K}{\partial \lambda_1}\right)_{04}^{(1)} = \sum_{N_1} \sum_{N_2} \frac{\lambda_1^{N_1}}{N_1!} \frac{\lambda_2^{N_2}}{N_2!} \int d\lambda_1' \dots \int d\lambda_{N_1+N_2}' \gamma_5' \dots \gamma_5^{N_1} \sum_{CT} \sum_{CT'} C \left(\begin{matrix} \lambda_1^{N_1} \\ \gamma_{m_1}' \dots \gamma_{m_{N_1}}' \end{matrix} \right) \left(\begin{matrix} \lambda_2^{N_2} \\ \gamma_{m_{N_1+1}}' \dots \gamma_{m_{N_1+N_2}}' \end{matrix} \right) X$$

Substituting (27) and (28) into (24) and using the formulas (12), we obtain:

$$\frac{\partial K_{N_0 P_0}}{\partial \lambda_1} = \int d\lambda_1' \gamma_5' K \left(\begin{matrix} \lambda_1^{N_1} \\ \gamma_{m_1}' \dots \gamma_{m_{N_1}}' \end{matrix} \right) \left(\begin{matrix} \lambda_2^{N_2} \\ \gamma_{m_{N_1+1}}' \dots \gamma_{m_{N_1+N_2}}' \end{matrix} \right) \left(\begin{matrix} \lambda_1^{N_1} \\ \gamma_{m_1}' \dots \gamma_{m_{N_1}}' \end{matrix} \right) + \left(C_{0,0}^{\lambda_1} + C_{0,2}^{(1)\lambda_1} \frac{p_0}{2} + C_{0,0}^{(1)\lambda_1} N_0 \right) K_{N_0 P_0} +$$

$$(29) \quad C_{0,2}^{(2)\lambda_1} \frac{\partial K_{N_0 P_0}}{\partial m_0^2} + C_{0,0}^{(2)\lambda_1} \frac{\partial K_{N_0 P_0}}{\partial m_f} + \left(C_{0,0}^{(1)\lambda_1} \frac{\lambda_1}{2} + C_{0,0}^{(1)\lambda_1} \lambda_1 + C_{0,0}^{(1)\lambda_1} \right) \frac{\partial K_{N_0 P_0}}{\partial \lambda_1} +$$

$$\left(2\lambda_4 C_{0,2}^{(1)\lambda_1} + C_{0,4}^{(1)\lambda_1} \right) \frac{\partial K_{N_0 P_0}}{\partial \lambda_4}$$

We can now go on as before, putting

$$K_{N_0 P_0} = A_{N_0 P_0} \tilde{K}_{N_0 P_0}$$

$$(30) \quad C^{\lambda_1} = 1 - \frac{\lambda_1}{2} C_{0,2}^{(1)\lambda_1} + C_{0,0}^{(1)\lambda_1} - C_{0,1}^{(1)\lambda_1}$$

$$\mathcal{D} = C^{\lambda_1} \frac{\partial}{\partial \lambda} - C_{0,2}^{(2)\lambda_1} \frac{\partial}{\partial m_0^2} - C_{0,0}^{(2)\lambda_1} \frac{\partial}{\partial m_f} + \left(2\lambda_4 C_{0,2}^{(1)\lambda_1} + C_{0,4}^{(1)\lambda_1} \right) \frac{\partial}{\partial \lambda_4}$$

we obtain:

$$\mathcal{D}(A_{N_0 P_0}) + A_{N_0 P_0} \mathcal{D} \tilde{K}_{N_0 P_0} =$$

$$(31) \quad = \left(C_{0,0}^{\lambda_1} + C_{0,2}^{(1)\lambda_1} \frac{p_0}{2} + C_{0,0}^{(1)\lambda_1} N_0 \right) K_{N_0 P_0} +$$

$$+ A_{N_0 P_0} \int d\lambda_1' \gamma_5' \tilde{K} \left(\begin{matrix} \lambda_1^{N_1} \\ \gamma_{m_1}' \dots \gamma_{m_{N_1}}' \end{matrix} \right) \left(\begin{matrix} \lambda_2^{N_2} \\ \gamma_{m_{N_1+1}}' \dots \gamma_{m_{N_1+N_2}}' \end{matrix} \right) \left(\begin{matrix} \lambda_1^{N_1} \\ \gamma_{m_1}' \dots \gamma_{m_{N_1}}' \end{matrix} \right)$$

that is:

$$(32) \quad \mathcal{D}(f A_{N_0 p_0}) = C_{q_0}^{\lambda} + C_{q_2}^{(1)\lambda} \frac{p_0}{2} + C_{i,0}^{(1)\lambda} N_0$$

$$(33) \quad A_{N_0 p_0} \mathcal{D} \tilde{K}_{N_0 p_0} = A_{N_0+1, p_0+1} \int d\{ \gamma_5^{\{ \tilde{K} \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \} / \{ t_1 \dots t_{p_0} \} \right.}$$

The formulas (32) and (33) become:

$$\mathcal{D}(f A) = C_{q_0}^{\lambda_1} ; \quad \mathcal{D}(f z_2) = C_{i,0}^{(1)\lambda_1} ; \quad \mathcal{D}(f z_3) = C_{q_2}^{(1)\lambda_1}$$

(34)

$$z_2^{-1} z_3^{-1/2} \mathcal{D} \tilde{K}_{N_0 p_0} = \int d\{ \gamma_5^{\{ \tilde{K} \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \} / \{ t_1 \dots t_{p_0} \} \right.}$$

where we have used the standard notation:

$$A_{N_0 p_0} = A z_2^{N_0} z_3^{\frac{p_0}{2}}$$

A change of parameters:

$$(35) \quad \begin{aligned} \lambda_1 &= \lambda_1 (\bar{\lambda}_1, \bar{m}_f, \bar{m}_n^2, \bar{\lambda}_4) \\ \lambda_4 &= \lambda_4 (\bar{\lambda}_1, \bar{m}_f, \bar{m}_n^2, \bar{\lambda}_4) \\ m_f &= m_f (\bar{\lambda}_1, \bar{m}_f, \bar{m}_n^2, \bar{\lambda}_4) \\ m_n^2 &= m_n^2 (\bar{\lambda}_1, \bar{m}_f, \bar{m}_n^2, \bar{\lambda}_4) \end{aligned}$$

gives:

$$\widetilde{K}(\lambda_1, m_1, m_2, \lambda_4) = \overline{K}(\bar{\lambda}_1, \bar{m}_1, \bar{m}_2, \bar{\lambda}_4).$$

and transforms the last equation (34) into

$$(36) \quad z_2^{-1} z_3^{-1} \left[\mathcal{D}(\bar{\lambda}) \frac{\partial}{\partial \bar{\lambda}} + \mathcal{D}(\bar{m}_1) \frac{\partial}{\partial \bar{m}_1} + \mathcal{D}(\bar{m}_2) \frac{\partial}{\partial \bar{m}_2} + \right. \\ \left. \mathcal{D}(\bar{\lambda}_4) \frac{\partial}{\partial \bar{\lambda}_4} \right] \overline{K}_{N, P_0} = \int d\{ \bar{K} \left(\begin{matrix} x_1, \dots, x_N \\ y_1, \dots, y_N \end{matrix} \right) | t_1, \dots, t_{P_0} \}$$

The theory will be renormalizable if a transformation (35) exists such that its inverse satisfies the equations:

$$(37) \quad \begin{aligned} \mathcal{D}(\bar{\lambda}_1) &= z_2 z_3^{\frac{1}{2}} \\ \mathcal{D}(\bar{m}_1) &= 0 \\ \mathcal{D}(\bar{m}_2) &= 0 \\ \mathcal{D}(\bar{\lambda}_4) &= 0 \end{aligned}$$

and the other equations obtained similarly from the derivatives of kernel with respect to other parameters. If this happens eq. (36)

becomes:

$$(38) \quad \frac{\partial \bar{K}}{\partial \lambda} = \int d\tau \delta_5^3 \bar{K} \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \tau t_1 \dots t_{p_0} \right)$$

and the derivatives of the kernel with respect to the other parameters will change similarly into the following expressions:

$$\frac{\partial \bar{K}}{\partial \bar{m}_1} = -i \int d\tau \bar{K} \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \tau t_1 \dots t_{p_0} \right) \quad 1)$$

(39)

$$\frac{\partial \bar{K}}{\partial \bar{m}_2} = -\frac{i}{2} \int d\tau \bar{K} \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \tau \tau t_1 \dots t_{p_0} \right)$$

$$\frac{\partial \bar{K}}{\partial \lambda_u} = \int d\tau \bar{K} \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \tau \tau \tau \tau t_1 \dots t_{p_0} \right)$$

1) The kernel for which the perturbative expansion is obtained by means of the substitution of "finite part" integrals into (17) is a formal solution of (39) if we assume that all f.p. integrals commute in a multiple integration. (See footnote at page 22).

4- To complete our knowledge of the system of differential equations which change the unrenormalized into renormalized parameters, we must compute also the derivatives of ^{the} kernels with respect to all the other parameters. By separating the convergent from the divergent parts we obtain:

$$\frac{\partial K_{H0t}}{\partial m_t} = -i \int d\zeta K^{(1)} \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \begin{matrix} \zeta \\ t_1 \dots t_{p_0} \end{matrix} \right) + \sum_{s,\sigma} \left(\frac{\partial K}{\partial m_t} \right)_{s,\sigma}^{(1)} \quad \text{where: } 3s + \sigma \leq 1$$

$$(40) \quad \frac{\partial K_{H0p_0}}{\partial m_t} = -\frac{i}{2} \int d\zeta K^{(1)} \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \begin{matrix} \zeta \\ t_1 \dots t_{p_0} \end{matrix} \right) + \sum_{s,\sigma} \left(\frac{\partial K}{\partial m_t^2} \right)_{s,\sigma}^{(1)} \quad \text{where: } 3s + \sigma \leq 2$$

$$\frac{\partial K_{H0p_0}}{\partial \lambda_4} = \int d\zeta K^{(1)} \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \begin{matrix} \zeta \\ t_1 \dots t_{p_0} \end{matrix} \right) + \sum_{s,\sigma} \left(\frac{\partial K}{\partial \lambda_4} \right)_{s,\sigma}^{(1)} \quad \text{where: } 3s + \sigma \leq 1$$

We define the "divergent cores":

$$C^{(m_1)} \left(\begin{matrix} \dot{u}_1 \dots \dot{u}_n \\ \dot{v}_1 \dots \dot{v}_n \end{matrix} \middle| \begin{matrix} \dot{w}_1 \dots \dot{w}_m \end{matrix} \right) = \int d\zeta L \left(\begin{matrix} \dot{u}_1 \dots \dot{u}_n \\ \dot{v}_1 \dots \dot{v}_n \end{matrix} \middle| \begin{matrix} \dot{w}_1 \dots \dot{w}_m \\ \zeta \end{matrix} \right)$$

$$C^{(m_2)} \left(\begin{matrix} \dot{u}_1 \dots \dot{u}_n \\ \dot{v}_1 \dots \dot{v}_n \end{matrix} \middle| \begin{matrix} \dot{w}_1 \dots \dot{w}_m \end{matrix} \right) = \int d\zeta L \left(\begin{matrix} \dot{u}_1 \dots \dot{u}_n \\ \dot{v}_1 \dots \dot{v}_n \end{matrix} \middle| \begin{matrix} \dot{w}_1 \dots \dot{w}_m \\ \zeta \end{matrix} \right)$$

$$(41) \quad C^{(\lambda_4)} \left(\begin{matrix} \dot{u}_1 \dots \dot{u}_n \\ \dot{v}_1 \dots \dot{v}_n \end{matrix} \middle| \begin{matrix} \dot{w}_1 \dots \dot{w}_m \end{matrix} \right) = \int d\zeta L \left(\begin{matrix} \dot{u}_1 \dots \dot{u}_n \\ \dot{v}_1 \dots \dot{v}_n \end{matrix} \middle| \begin{matrix} \dot{w}_1 \dots \dot{w}_m \\ \zeta \end{matrix} \right)$$

and have:

$$\left(\frac{\partial K}{\partial m_t} \right)_{s,\sigma}^{(1)} = C_{0,0}^{m_t} K_{H0p_0}; \quad \left(\frac{\partial K}{\partial m_t^2} \right)_{s,\sigma}^{(1)} = C_{0,2}^{m_t} \frac{2}{m_t^2} K_{H0p_0}; \quad \left(\frac{\partial K}{\partial m_t} \right)_{s,\sigma}^{(1)} = C_{1,0}^{m_t} \frac{\partial K}{\partial \lambda_4}$$

(42)

$$\begin{aligned}
 \left(\frac{\partial K}{\partial m_0^2} \right)_{q_0}^{(1)} &= c_{q_0}^{m_0^2} K_{N_0 P_0} ; & \left(\frac{\partial K}{\partial m_0^2} \right)_{q_2}^{(1)} &= c_{q_2}^{m_0^2} \frac{\partial K_{N_0 P_0}}{\partial m_0^2} \\
 \left(\frac{\partial K}{\partial \lambda_{10}} \right)^{(1)} &= c_{q_0}^{\lambda_4} K_{N_0 P_0} ; & \left(\frac{\partial K}{\partial \lambda_4} \right)_{11}^{(1)} &= c_{1,1}^{\lambda_4} \frac{\partial K_{N_0 P_0}}{\partial \lambda_1} ; & \left(\frac{\partial K}{\partial \lambda_{10}} \right)^{(2)} &= N_0 c_{1,0}^{(1)\lambda_4} K_{N_0 P_0} + \lambda_1 c_{1,0}^{(1)\lambda_4} \frac{\partial K_{N_0 P_0}}{\partial \lambda_1} + c_{1,0}^{(2)\lambda_4} \frac{\partial K_{N_0 P_0}}{\partial \lambda_1} \\
 \left(\frac{\partial K}{\partial \lambda_4} \right)_{214}^{(1)} &= c_{q_4}^{\lambda_4} \frac{\partial K_{N_0 P_0}}{\partial \lambda_4} ; & \left(\frac{\partial K}{\partial \lambda_4} \right)_{212}^{(1)} &= \frac{P_0}{2} c_{q_2}^{(1)\lambda_4} K_{N_0 P_0} + \frac{\lambda_1}{2} c_{q_2}^{(1)\lambda_4} \frac{\partial K_{N_0 P_0}}{\partial \lambda_1} + 2\lambda_4 c_{q_2}^{(1)\lambda_4} \frac{\partial K_{N_0 P_0}}{\partial \lambda_4} + c_{q_2}^{(2)\lambda_4} \frac{\partial K_{N_0 P_0}}{\partial m_0^2}
 \end{aligned}$$

where we have used the conditions:

$$\begin{aligned}
 c_{q_2}^{(m_f)}(\dot{u}_1, \dot{u}_2) &= c_{q_2}^{m_f} \frac{\partial [H_1, H_2]}{\partial m_0^2} ; & c_{1,0}^{(m_f)}(\dot{u}_1) &= c_{1,0}^{m_f} \frac{\partial (u, \dot{u})}{\partial m_f} \\
 c_{q_2}^{(\lambda_4)}(\dot{u}_1, \dot{u}_2) &= c_{q_2}^{(1)\lambda_4} [H_1, H_2] + c_{q_2}^{(2)\lambda_4} \frac{\partial [H_1, H_2]}{\partial m_0^2} ; & c_{1,0}^{(\lambda_4)}(\dot{u}_1) &= c_{1,0}^{\lambda_4} \int d^3x \delta^3(x) \delta^3(\dot{x}) \\
 (43) \quad c_{q_4}^{(\lambda_4)}(\dot{u}_1, \dot{u}_2, \dot{u}_3, \dot{u}_4) &= c_{q_4}^{\lambda_4} \int d^3x [\dot{u}_1, \dot{u}_2, \dot{u}_3, \dot{u}_4] ; & c_{1,0}^{(\lambda_4)}(\dot{u}_1) &= c_{1,0}^{(1)\lambda_4} (u_1, v_1) + c_{1,0}^{(2)\lambda_4} \frac{\partial u}{\partial m_f} \\
 c_{q_2}^{(m_0^2)}(\dot{u}_1, \dot{u}_2) &= c_{q_2}^{m_0^2} \frac{\partial [H_1, H_2]}{\partial m_0^2}
 \end{aligned}$$

The equation (40) becomes thus:

$$\begin{aligned}
 \mathcal{D}' K_{N_0 P_0} &= c_{q_0}^{m_f} K_{N_0 P_0} - i \int d^3x K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \dot{z} \middle| t_1 \dots t_{P_0} \right) \\
 (44) \quad \mathcal{D}'' K_{N_0 P_0} &= c_{q_0}^{m_0^2} K_{N_0 P_0} - \frac{i}{2} \int d^3x K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \dot{z} \dot{z} \middle| t_1 \dots t_{P_0} \right) \\
 \mathcal{D}''' K_{N_0 P_0} &= \left(c_{q_0}^{\lambda_4} + N_0 c_{1,0}^{(1)\lambda_4} + \frac{P_0}{2} c_{q_2}^{(1)\lambda_4} \right) K_{N_0 P_0} + \int d^3x K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \dot{z} \dot{z} \dot{z} \middle| t_1 \dots t_{P_0} \right)
 \end{aligned}$$

where:

$$\begin{aligned}
 \mathcal{D}' &= c^{mf} \frac{\partial}{\partial m_f} - c_{a_2}^{mf} \frac{\partial}{\partial m_a^2} \quad ; \quad c^{mf} = 1 - c_{a_2}^{mf} \\
 (45) \quad \mathcal{D}'' &= c^{mb} \frac{\partial}{\partial m_b^2} \quad ; \quad c_m^b = 1 - c_{a_2}^{mb} \\
 \mathcal{D}''' &= c^{\lambda_4} \frac{\partial}{\partial \lambda_4} - c_{a_2}^{(2)\lambda_4} \frac{\partial}{\partial m_a^2} - c_{a_2}^{(2)\lambda_4} \frac{\partial}{\partial m_f} - \left(\frac{\lambda_1}{2} c_{a_2}^{(1)\lambda_4} + \lambda_1 c_{a_2}^{(1)\lambda_4} + c_{a_2}^{(1)\lambda_4} \right) \frac{\partial}{\partial \lambda_1} \\
 c^{\lambda_4} &= 1 - 2\lambda_4 c_{a_2}^{(1)\lambda_4} - c_{a_2}^{\lambda_4}
 \end{aligned}$$

Proceeding as before we obtain easily the other equations of the transformation from unrenormalized to renormalized parameters.

From the first of the (45) we have:

$$\begin{aligned}
 \mathcal{D}'(m_f) &= z_2 \\
 \mathcal{D}'(\bar{m}_a^2) &= 0 \\
 (46) \quad \mathcal{D}'(\bar{\lambda}_1) &= 0 \\
 \mathcal{D}'(\bar{\lambda}_4) &= 0
 \end{aligned}$$

from the second:

$$\begin{aligned}
 \mathcal{D}''(\bar{m}_f) &= 0 \\
 \mathcal{D}''(\bar{m}_b^2) &= z_3 \\
 (47) \quad \mathcal{D}''(\bar{\lambda}_1) &= 0 \\
 \mathcal{D}''(\bar{\lambda}_4) &= 0
 \end{aligned}$$

and from the last:

$$\begin{aligned}
 (48) \quad & \mathcal{D}'''(\bar{m}_f) = 0 \\
 & \mathcal{D}'''(m_b^2) = 0 \\
 & \mathcal{D}'''(\bar{\lambda}_1) = 0 \\
 & \mathcal{D}'''(\bar{\lambda}_4) = z_3^2
 \end{aligned}$$

We can conclude, keeping in mind the equations (37), (46), (47) and (48), that the theory is renormalizable if the following system is compatible:

$$(49) \quad \left\{ \begin{array}{l} \mathcal{D}(\bar{\lambda}_1) = z_2 z_3^{\frac{1}{2}} \\ \mathcal{D}'(\bar{\lambda}_1) = 0 \\ \mathcal{D}''(\bar{\lambda}_1) = 0 \\ \mathcal{D}'''(\bar{\lambda}_1) = 0 \end{array} \right\} \left\{ \begin{array}{l} \mathcal{D}(\bar{m}_f) = 0 \\ \mathcal{D}'(\bar{m}_f) = z_2 \\ \mathcal{D}''(\bar{m}_f) = 0 \\ \mathcal{D}'''(m_b^2) = 0 \end{array} \right\} \left\{ \begin{array}{l} \mathcal{D}(\bar{m}_b^2) = 0 \\ \mathcal{D}'(\bar{m}_b^2) = 0 \\ \mathcal{D}''(\bar{m}_b^2) = z_3 \\ \mathcal{D}'''(\bar{m}_b^2) = 0 \end{array} \right\} \left\{ \begin{array}{l} \mathcal{D}(\bar{\lambda}_4) = 0 \\ \mathcal{D}'(\bar{\lambda}_4) = 0 \\ \mathcal{D}''(\bar{\lambda}_4) = 0 \\ \mathcal{D}'''(\bar{\lambda}_4) = z_3^2 \end{array} \right\}$$

The conditions of compatibility are:

$$\begin{aligned}
 & 1 - c_{40}^{m_b^2} \neq 0 \qquad \qquad \qquad 1 - c_{12}^{m_b^2} \neq 0 \\
 (50) \quad & \left(1 - \frac{\lambda_1}{2} c_{12}^{(1)\lambda_1} - c_{10}^{(1)\lambda_1} \lambda_1 - c_{11}^{\lambda_1} \right) \left(1 - 2 \lambda_4 c_{12}^{(1)\lambda_4} + c_{14}^{\lambda_4} \right) + \\
 & - \left(2 \lambda_4 c_{12}^{(1)\lambda_1} + c_{14}^{\lambda_1} \right) \left(\frac{\lambda_1}{2} c_{12}^{(1)\lambda_4} + c_{10}^{(1)\lambda_4} + c_{11}^{\lambda_4} \right) \neq 0
 \end{aligned}$$

These conditions are generally verified except for particular values of the parameters which would make all the relations to (50) vanish

III Scalar neutral theory

We start from the Hamiltonian:

$$H = \int \bar{\psi} + \psi :$$

we can define the kernel $K_{\alpha\beta}$ and its derivative with respect to $\lambda = -ig$.

After separating the divergent from ^{the}convergent parts we obtain an equation like (8). In this case the sum is over the terms $\left(\frac{\partial K}{\partial \lambda}\right)_{s,r}$ with

$S = 0$	$\sigma = 0$
$S = 1$	$\sigma = 0$
$S = 0$	$\sigma = 2$
$S = 0$	$\sigma = 4$
$S = 1$	$\sigma = 1$

which we have already discussed, and over the divergent term ¹⁾ with

$$S = 0 \quad \sigma = 3 \quad (51)$$

also.

Following the same pattern as that ^{of the}p.s. theory we can easily show that the scalar theory is renormalizable ²⁾, provided two now

-
- 1) The divergent term $S=0, \sigma=1$ must be put equal to zero if $(\{ \})=0$. If instead we put $(\{ \}) = f(m)$; $[\{ \}] = g(m)$ the case $S=0, \sigma=1$ does not lead to new terms in the Hamiltonian but only affects the renormalization coefficient.
 - 2) This occurs only if all f.p. integrals commute in a multiple integration, whether the integrand is a symmetric function of its

terms are added to the Hamiltonian, which becomes:

$$(52) \quad H = g_1 \bar{\psi} \psi + g_2 \psi^3 + g_3 \psi^4$$

The kernel is the following:

$$(53) \quad K_{N_0 P_0} = \sum_{S_1=0}^{\infty} \sum_{S_2=0}^{\infty} \sum_{S_3=0}^{\infty} \frac{\lambda_1^{S_1}}{S_1!} \frac{\lambda_2^{S_2}}{S_2!} \frac{\lambda_3^{S_3}}{S_3!} \int d\zeta_1 \dots \int d\zeta_N$$

$$\begin{pmatrix} x_1 \dots x_{N_0} & \zeta_1 \dots \zeta_{S_1} \\ y_1 \dots y_{N_0} & \zeta_1 \dots \zeta_{S_1} \end{pmatrix} \left[t_1 \dots t_{P_0} \zeta_1 \dots \zeta_{S_1} \zeta_{S_1+1} \zeta_{S_1+2} \dots \zeta_{S_1+S_2} \zeta_{S_1+S_2+1} \zeta_{S_1+S_2+2} \dots \zeta_{S_1+S_2+S_3} \zeta_{S_1+S_2+S_3+1} \zeta_{S_1+S_2+S_3+2} \dots \right]$$

The sums are over values S_1 and S_3 for which $S_1 + 3S_3$ has the same parity as P_0 .

The derivatives of the kernel with respect to the parameters are given by:

$$\frac{\partial K_{N_0 P_0}}{\partial \lambda_1} = \int d\zeta \, K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \zeta_1 \dots \zeta_{S_1} \right)$$

$$(54) \quad \frac{\partial K_{N_0 P_0}}{\partial \lambda_4} = \int d\zeta \, K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \zeta_1 \dots \zeta_{S_1} \right)$$

(cont. of footnote N° 1) of the preceding page)

variables or not. In our past work (E.R. Caianiello, N. Cimento, 14, 185 (1959) end of sect. 4.5) it did not seem legitimate to assume this property a priori, hence it was concluded that this theory is not renormalizable. More recent investigations⁷⁾, however, have given assurance that such inversion is always legitimate in the cases of physical interest. We are thus now in complete agreement with the current opinion on renormalizable theories, at least as far as necessary conditions are concerned.

$$\frac{\partial K}{\partial A_3} = \int d\tau K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \tau \tau t_1 \dots t_{p_0} \right)$$

$$\frac{\partial K}{\partial A_4} = -i \int d\tau K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \dot{\tau} \dot{\tau} t_1 \dots t_{p_0} \right)$$

$$\frac{\partial K}{\partial m_0} = -\frac{i}{2} \int d\tau K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \ddot{\tau} \ddot{\tau} t_1 \dots t_{p_0} \right)$$

where:

$$A_1 = -ig_1 \quad A_3 = -ig_2 \quad A_4 = -ig_3$$

IV. - The charged mesonic field.

The formal study of the charged theory follows the same pattern as that of the neutral theory. Let us consider the Hamiltonian of the charged pseudoscalar theory:

$$(55) \quad H = g_1 \bar{\psi} \gamma_\mu \psi \varphi^\mu + g_2 (\varphi_\mu \varphi^\mu)^2 \quad \mu=1,2,3$$

where:

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Keeping in mind the relations:

$$(56) \quad \begin{array}{llll} \gamma_1 \gamma_\mu = \beta_\mu & \gamma_2 \gamma_\mu = i \beta_\mu & \gamma_3 \gamma_\mu = \gamma_\mu & \gamma_\mu \beta_\mu = 0 \\ \gamma_1 \gamma_\mu = \beta_\mu & \gamma_2 \gamma_\mu = -i \beta_\mu & \gamma_3 \gamma_\mu = -\gamma_\mu & \gamma_\mu \beta_\mu = 0 \end{array}$$

where

$$\beta_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \beta_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the form (55) of the Hamiltonian one finds that the kernel and its derivatives have the expressions:

$$K_{N_0 P_0} = \sum_{s_1} \sum_{s_2} \frac{\lambda_1^{s_1}}{s_1!} \frac{\lambda_2^{s_2}}{s_2!} \int d\beta_1 \dots \int d\beta_N \sum_{p_1 \dots p_N} \gamma_5^{s_1} \dots \gamma_5^{s_2} \gamma_{p_1} \dots \gamma_{p_N}$$

$$\left(\begin{matrix} x_1 \dots x_{N_0} & z_1 \dots z_{s_1} \\ y_1 \dots y_{N_0} & z_1 \dots z_{s_1} \end{matrix} \right) \left[\begin{matrix} t_1 \dots t_{p_1} & z_1 \dots z_{s_1} & z_{s_1+1} \dots z_{s_1+p_1} \\ t_1 \dots t_{p_1} & z_1 \dots z_{s_1} & z_{s_1+1} \dots z_{s_1+p_1} \end{matrix} \right]$$

$$\frac{\partial K_{N_0 P_0}}{\partial \lambda_1} = \int d\beta \gamma_{p_1} \gamma_5^3 K \left(\begin{matrix} x_1 \dots x_{N_0} & z \\ y_1 \dots y_{N_0} & z \end{matrix} \middle| t_1 \dots t_{p_0} \right)$$

(57)

$$\frac{\partial K_{N_0 P_0}}{\partial m_f} = -i \int d\beta K \left(\begin{matrix} x_1 \dots x_{N_0} & z \\ y_1 \dots y_{N_0} & z \end{matrix} \middle| t_1 \dots t_{p_0} \right)$$

$$\frac{\partial K_{N_0 P_0}}{\partial m_0^2} = -\frac{1}{2} \int d\beta K \left(\begin{matrix} x_1 \dots x_{N_0} & z \\ y_1 \dots y_{N_0} & z \end{matrix} \middle| z z t_1 \dots t_{p_0} \right)$$

$$\frac{\partial K_{N_0 P_0}}{\partial \lambda_4} = \int d\beta K \left(\begin{matrix} x_1 \dots x_{N_0} & z \\ y_1 \dots y_{N_0} & z \end{matrix} \middle| z z z z t_1 \dots t_{p_0} \right)$$

where $\lambda_1 = -ig_1$ $\lambda_4 = -ig_2$

$$(58) \quad \{z, z\} = (\gamma_p + \gamma_n) \{z, z\}_N$$

$$[z, z_j] = \delta_{p, p_j} [z, z_j]_\pi$$

$\{z, z\}_N$ and $[z, z_j]_\pi$ are the free propagators for nucleon and pion respectively; $(\tau_p + \tau_n) (\xi_i, \xi_j)$

is the g -dimensional matrix

$$\begin{pmatrix} (\xi_i, \xi_j)_N & 0 \\ 0 & (\xi_i, \xi_j)_\pi \end{pmatrix}$$

Let us separate in the second of the equations (57) the divergent from the convergent part, using the same notation as in the neutral case;

we obtain:

$$(59) \quad \frac{\partial K}{\partial \lambda_1} = \int d\tau \gamma_5^2 \gamma^2 K^{(b)} \left(\begin{matrix} x_1, \dots, x_{N_0} \\ y_1, \dots, y_{N_0} \end{matrix} \middle| \tau t_1, \dots, \tau t_{N_0} \right) + \sum_{s_1} \left(\frac{\partial K}{\partial \lambda} \right)_{s_1, \sigma}^{(2)}$$

We can introduce the divergent cores:

$$(60) \quad C^{(b)} \left(\begin{matrix} \dot{u}_1, \dots, \dot{u}_n \\ \dot{v}_1, \dots, \dot{v}_n \end{matrix} \middle| \dot{w}_1, \dots, \dot{w}_m \right) = \int d\tau \gamma_5^2 \gamma^2 L \left(\begin{matrix} \dot{u}_1, \dots, \dot{u}_n \\ \dot{v}_1, \dots, \dot{v}_n \end{matrix} \middle| \dot{w}_1, \dots, \dot{w}_m \right)$$

where

$$(61) \quad L \left(\begin{matrix} \dot{u}_1, \dots, \dot{u}_n \\ \dot{v}_1, \dots, \dot{v}_n \end{matrix} \middle| \dot{w}_1, \dots, \dot{w}_m \right) = \sum_{s_1} \sum_{s_2} \frac{\lambda_1^{s_1}}{s_1!} \frac{\lambda_2^{s_2}}{s_2!} \mathcal{D}_1^{N-1} \sum_{\mu_1, \dots, \mu_N} \tau_{\mu_1} \dots \tau_{\mu_N} \gamma_5^{s_1+s_2} \times \\ \left(\begin{matrix} \dot{u}_1, \dots, \dot{u}_n \\ \dot{v}_1, \dots, \dot{v}_n \end{matrix} \middle| \tau_1, \dots, \tau_{s_1} \right) \left[\begin{matrix} \dot{w}_1, \dots, \dot{w}_m \\ \tau_{s_1+1}, \dots, \tau_{s_1+s_2} \end{matrix} \right] \tau_{s_1+s_2+1} \dots \tau_{s_1+s_2+m}$$

These cores satisfy the equations (12) where the constants C do not contain the matrices τ_i explicitly. To see this let us consider core:

$$(62) \quad C \left(\begin{matrix} x \\ y \end{matrix} \right) = \int d\tau \gamma_5^2 \tau^2 \sum_{s_1} \sum_{s_2} \frac{\lambda_1^{s_1}}{s_1!} \frac{\lambda_2^{s_2}}{s_2!} \mathcal{D}_2^{N-1} \sum_{\mu_1, \dots, \mu_N} \tau_{\mu_1} \dots \tau_{\mu_N} \gamma_5^{s_1+s_2} \times \\ \left(\begin{matrix} x \\ \tau_1, \dots, \tau_{s_1} \end{matrix} \right) \left[\begin{matrix} y \\ \tau_{s_1+1}, \dots, \tau_{s_1+s_2} \end{matrix} \right] \tau_{s_1+s_2+1} \dots \tau_{s_1+s_2+m}$$

where the first sum is over all indices s_i having odd parity, and examine the n^{th} term of its perturbative expansion:

$$(63) \quad \int d\tau \frac{\lambda_1^{s_1}}{s_1!} \frac{\lambda_2^{s_2}}{s_2!} \mathcal{D}_2^{N-1} \sum_{\mu_1, \dots, \mu_N} \sum_{i_1, i_2} \tau_{\mu_{i_1}} \gamma_5^{i_1} (\tau_{i_1} \tau_{i_2}) \tau_{\mu_{i_2}} \gamma_5^{i_2} \dots \times \\ \sum' [\tau_{\mu_1} x] [\tau_{\mu_2} y] \dots [\tau_{\mu_{N-1}} \tau_{\mu_N}] \dots$$

(\sum means summation over all the permutations i_1, \dots, i_{s+2} of $012 \dots s$. P is the parity of the permutation

\sum' means summation over all the permutations i_1, \dots, i_{s+2} of $0, 1, \dots, N$ which satisfy the limitations:

$$i_{m_{s+1}} < i_{m_{s+2}} \quad i_{m_1} < i_{m_2} < i_{m_3} \dots$$

Keeping in mind (58) and the commutation relations of matrices γ_i , the expression (63) becomes:

$$(64) \int d\gamma \frac{1}{S_1} \frac{1}{S_2} \mathcal{D}^{N+1} \sum_i (-1)^P K_{im} \int_{P \times P} (\tilde{T}_P + \tilde{T}_N) (\gamma_i, \gamma_m) \delta^5 \dots$$

$$(\gamma_{i_{m_1}}, \gamma_{i_{m_2}}) \delta^5 \sum' [\gamma_{m_1}, \gamma_{m_2}] - [\gamma_{m_{s+1}}, \gamma_{m_{s+2}}]$$

where K_{im} is a numerical coefficient which depends on the matrices γ_i and changes when the permutation changes.

Hence like in the neutral case we have:

$$C_{(X,Y)}^{(1)\lambda} = C_{0,1}^{(1)\lambda} [u_1, u_2] + C_{0,2}^{(2)\lambda} \frac{\gamma[u_1, u_2]}{\gamma m_1^2}$$

where $C_{0,1}^{(1)\lambda}$ and $C_{0,2}^{(2)\lambda}$ do not contain matrices γ_i . We arrive at the same result if we consider $C_{0,1}^{(2)\lambda}, C_{0,2}^{(3)\lambda}, \dots$ and so on.

Hence we can go on as in the previous case and obtain an equation formally equal to the equ. (36), the only difference is in the numerical value of the constants which are contained in the \mathcal{D} .

In the usual way we can examine the derivatives of respect to parameters $\lambda_{i,m}$ and so we obtain a system like (49). By extending to this system the considerations made in the previous case we conclude that the charged P.S. theory satisfies the necessary con-

ditions for its renormalizability.

What we have proved is true for the scalar charged theory also.
we obtain the expression for the kernel ^{by} eliminating the τ_i matrices
from (57).

This work was in part performed while one of the Authors
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A P P E N D I X

Consider two meson fields φ_1 and φ_2 (it does not matter whether with the same or different mass), coupled by a term

$$(1A) \quad H = g \varphi_1^2 \varphi_2^2$$

We get for the kernel the expression:

$$(2A) \quad K_{p_0 p'_0} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int d\tau_1 \dots \int d\tau_k [t_1 \dots t_{p_0} \tau_1 \tau_1 \tau_2 \tau_2 \dots \tau_k \tau_k] \times \\ [t'_1 \dots t'_{p'_0} \tau_1 \tau_1 \dots \tau_k \tau_k]$$

$$\lambda = -ig$$

Consider:

$$(3A) \quad \frac{\partial K_{p_0 p'_0}}{\partial \lambda} = \int d\tau \, K(t_1 \dots t_{p_0} \tau \tau / t'_1 \dots t'_{p'_0} \tau \tau)$$

The λ -derivative of the kernel can be written:

$$(4A) \quad \frac{\partial K_{p_0 p'_0}}{\partial \lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int d\tau \, \hat{\mathcal{D}}_1^{k-1} K^{(1)}(t_1 \dots t_{p_0} \tau \tau / t'_1 \dots t'_{p'_0} \tau \tau)$$

The divergent part is separated as usual:

$$(5A) \quad \frac{\partial K}{\partial \lambda} = \int d\{ \} K^{(3)}(t_1 \dots t_{p_0} \{ \} | t'_1 \dots t'_{p'_0} \{ \}) + \sum_{s,r} \left(\frac{\partial K}{\partial \lambda} \right)_{s,r}^{(3)}$$

where we have used the expression:

$$(6A) \quad \frac{\partial K}{\partial \lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int d\{ \} \hat{\mathcal{D}}_1^{k-1} \sum_{s=0}^{k-1} \sum_{v=0}^{k-1} [\tau_{e_1} \dots \tau_{e_{s+1}}] [\tau_{e'_1} \dots \tau_{e'_{v+1}}] \times \\ K^{(k)}(\tau_{e_1} \dots \tau_{e_s} t_1 \dots t_{p_0} | \tau_{e'_1} \dots \tau_{e'_v} t'_1 \dots t'_{p'_0})$$

Divergences can actually arise only if $s + \sigma \leq 4$

and furthermore s and σ can have only even values. Therefore

$$(7A) \quad \frac{\partial K}{\partial \lambda} = \int d\{ \} K^{(3)}(t_1 \dots t_{p_0} \{ \} | t'_1 \dots t'_{p'_0} \{ \}) + \left(\frac{\partial K}{\partial \lambda} \right)_{00}^{(3)} + \left(\frac{\partial K}{\partial \lambda} \right)_{02}^{(3)} + \left(\frac{\partial K}{\partial \lambda} \right)_{20}^{(3)} + \left(\frac{\partial K}{\partial \lambda} \right)_{40}^{(3)} + \left(\frac{\partial K}{\partial \lambda} \right)_{22}^{(3)}$$

The divergent core is

$$(8A) \quad c^{(\lambda)}(\tau_{e_1} \dots \tau_{e_s} | \tau_{e'_1} \dots \tau_{e'_{\sigma}}) = \int d\{ \} \sum_{M=1}^{\infty} \frac{\lambda^M}{M!} \mathcal{D}_1^{M-1} [\tau_{e_1} \dots \tau_{e_s} \{ \} \tau_{e'_1} \dots \tau_{e'_{\sigma}} \{ \}] \times \\ [\tau_{e_1} \dots \tau_{e_s} \{ \} \tau_{e'_1} \dots \tau_{e'_{\sigma}} \{ \}]$$

So that

$$\left(\frac{\partial K}{\partial \lambda} \right)_{00}^{(3)} = c_{00}^{(\lambda)} K_{p_0 p'_0} \quad \left(\frac{\partial K}{\partial \lambda} \right)_{22}^{(3)} = c_{22}^{(\lambda)} \frac{\partial K_{p_0 p'_0}}{\partial \lambda}$$

$$\left(\frac{\partial K}{\partial \lambda} \right)_{02}^{(3)} = c_{02}^{(1)\lambda} \frac{p'_0}{2} K_{p_0 p'_0} + \lambda c_{02}^{(11)\lambda} \frac{\partial K_{p_0 p'_0}}{\partial \lambda} + c_{02}^{(2)\lambda} \frac{\partial^2 K_{p_0 p'_0}}{\partial \lambda^2}$$

$$\left(\frac{\partial K}{\partial \lambda} \right)_{20}^{(3)} = c_{20}^{(1)\lambda} \frac{p_0}{2} K_{p_0 p'_0} + \lambda c_{20}^{(11)\lambda} \frac{\partial K_{p_0 p'_0}}{\partial \lambda} + c_{20}^{(2)\lambda} \frac{\partial^2 K_{p_0 p'_0}}{\partial \lambda^2}$$

$$(9A) \quad \left(\frac{\partial K}{\partial \lambda} \right)_{0,4}^{(3)} = c_{0,4}^{(\lambda)} \int d\{ \} K(t_1 \dots t_{p_0} | t'_1 \dots t'_{p'_0} \{ \})$$

$$\left(\frac{\partial K}{\partial \lambda} \right)_{4,0}^{(3)} = c_{4,0}^{(\lambda)} \int d\{ \} K(t_1 \dots t_{p_0} \{ \} | t'_1 \dots t'_{p'_0})$$

(7A) becomes thus:

$$\begin{aligned} \frac{\partial K}{\partial \lambda} &= \int d\vec{z} K^{(3)}(t_1 \dots t_{p_0} \vec{z} \vec{z} / t'_1 \dots t'_{p'_0} \vec{z} \vec{z}) + \left(C_{g_0}^\lambda + C_{g_2}^{(1)\lambda} \frac{p_0}{2} + C_{g_0}^{(1)\lambda} \frac{p_0}{2} \right) K_{p_0 p'_0} \\ (10A) &+ \left(\lambda C_{g_0,2}^{(1)\lambda} + \lambda C_{g_2,0}^{(1)\lambda} + C_{g_2,2}^\lambda \right) \frac{\partial K}{\partial \lambda} + \left(C_{g_2,0}^{(2)\lambda} + C_{g_2,2}^{(1)\lambda} \right) \frac{\partial K}{\partial m_0^2} + \\ &+ C_{g_4}^\lambda \int d\vec{z} K(t_1 \dots t_{p_0} \vec{z} \vec{z} \vec{z} \vec{z} / t'_1 \dots t'_{p'_0} \vec{z} \vec{z}) + C_{g_4,0}^\lambda \int d\vec{z} K(t_1 \dots t_{p_0} \vec{z} \vec{z} \vec{z} \vec{z} / t'_1 \dots t'_{p'_0} \vec{z} \vec{z}) \end{aligned}$$

which yields, with our customary treatment

$$\begin{aligned} \frac{\partial \bar{K}}{\partial \lambda} &= \int d\vec{z} K(t_1 \dots t_{p_0} \vec{z} \vec{z} / t'_1 \dots t'_{p'_0} \vec{z} \vec{z}) + \\ (11A) &+ C_{g_4}^\lambda \int d\vec{z} K(t_1 \dots t_{p_0} \vec{z} \vec{z} \vec{z} \vec{z} / t'_1 \dots t'_{p'_0} \vec{z} \vec{z}) + C_{g_4,0}^\lambda \int d\vec{z} K(t_1 \dots t_{p_0} \vec{z} \vec{z} \vec{z} \vec{z} / t'_1 \dots t'_{p'_0} \vec{z} \vec{z}) \end{aligned}$$

i.e., by comparison with (3-A), covariance is not secured.

It would be straightforward to verify that, if together with (1A) two more terms are added to the interaction,

$$(12A) \quad H = g_1 \varphi_1^2 \varphi_2^2 + g_1 \varphi_1^4 + g_2 \varphi_2^4,$$

then two more derivative equations must be considered, and covariance can be obtained (together with the equations of the renormalization group).

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